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The Riccati Equation $y' = x + y^2$ for an Airy Function

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INTRODUCTION: ITERATIVE PROCESS

The general Riccati equation [1]

$$y' = a(x) + b(x)y + c(x)y^2$$

admits a simple iterative process. Assuming that

$$y = u + y_1$$

and determining u from the linear equation

$$u' = a + bu,$$

we see that the residual y_1 also satisfies the general Riccati equation

$$y' = a_1 + b_1 y_1 + c_1 y_1^2$$

with the new coefficients depending on x:

$$a_1(x) = c(x)u^2(x),$$

$$b_1(x) = b(x) + 2c(x)u(x),$$

$$c_1(x) = c(x).$$

The dependence on x is not specified explicitly.

Continuing this process, we obtain a series for the solution y with a remainder y_n:

$$y = u + u_1 + u_2 + \dots + u_{n-1} + y_n.$$

The terms of this series are the solutions to the linear equations

$$u' = a_n + b_n u_n.$$

The coefficients of the equations are determined by the recurrence formulas

$$a_{n+1} = cu_n^2,$$

$$b_{n+1} = b_n + 2cu_n.$$

An important feature of this formal series is the simplicity of the transition $n \to n + 1$, which is achieved at the cost of solving a linear differential equation at every iteration step. Clearly, this procedure is suited for modern computers.

Lemma (on a bounded solution). *If the linear equation*

$$u' = a(x) + b(x)u$$

has complex coefficients such that $a(x)$ is bounded,

$$|a(x)| \le a,$$

and $b(x)$ belongs to the right half-plane, i.e.,

$$\mathrm{Re}\{b(x)\} \ge p > 0,$$

then this equation has the particular solution

$$u(x) = -\int_x^{\infty} a(s)\exp\left\{-\int_x^s b(r)dr\right\}ds$$

bounded on the entire line:

$$|u(x)| \le u = \frac{a}{p}.$$

This inequality becomes an equality if and only if $a(x)$ and $b(x)$ are constants. The bounded solution found is unique. Any other solution is unbounded when $x \to \infty$.

The sufficient conditions for convergence are easily derived by estimating the following two positive numbers at every iteration step:

$$a_n = \max_x |a_n(x)|, \quad p_n = \min_x \mathrm{Re}\{b_n(x)\}.$$

From the recurrence formulas and the estimate

$$|u_n(x)| \le \frac{a_n}{p_n}$$

it is easy to obtain the inequalities

$$a_{n+1} \le c\left(\frac{a_n}{p_n}\right)^2, \quad p_{n+1} \ge p_n - \frac{2a_n c}{p_n}.$$

By introducing the dimensionless parameter

$$\theta_n = \frac{4a_n c}{p_n^2},$$

this chain of inequalities becomes much clearer:

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$$\theta_{n+1} \le \left(\frac{\theta_n}{2-\theta_n}\right)^2,$$

$$a_{n+1} \le \frac{\theta_n}{4} a_n,$$

$$p_{n+1} \ge \left(1 - \frac{\theta_n}{2}\right) p_n.$$

The inequality for θ generates a countable set of sufficient conditions for convergence. Indeed, if

$$\theta_n \le 1,$$

then the next θ_{n+1} and all of the subsequent θ_{n+k} satisfy the same inequality

$$\theta_{n+k} \le 1.$$

Hence, starting from a sufficiently large n, we have

$$|u_n(x)| \le \frac{1}{2^n} u;$$

i.e., the series converges more rapidly than a geometric progression with a ratio of $\frac{1}{2}$.

However, if we have the strict inequality

$$\theta_n < 1,$$

then convergence is much faster and is the same as in Newton's method; i.e., the number of correct digits is doubled at every iteration step. The above analysis can be extended to the equation

$$my' = a + by + y^2.$$

This equation differs from that examined above by the positive factor m multiplying the derivative y', which is equivalent to variations in x. Therefore, for this equation, the above conditions for the convergence of the iterative process also hold as formulated in terms of the properties of $a(x)$, $b(x)$, and $p(x)$:

$$a_n = \max_x |a_n(x)|, \quad p_n = \min_x \operatorname{Re}\{b_n(x)\}.$$

For constant coefficients, the process converges to the root of the quadratic equation

$$a + by + y^2 = 0.$$

The terms of the series are precisely calculated for any n:

$$u_n = -\frac{a_n}{b_n},$$

and we obtain a chain of recurrence formulas

$$a_{n+1} = c_n\left(\frac{a_n}{b_n}\right)^2, \quad b_{n+1} = b_n - \frac{2a_n c_n}{b_n}, \quad c_{n+1} = c_n$$

for solving the quadratic equation. Introducing the dimensionless (complex) parameter $\theta_n = \frac{4a_n c}{b_n^2}$, we again obtain for θ_n the one-dimensional mapping

$$\theta_{n+1} = \left(\frac{\theta_n}{2-\theta_n}\right)^2,$$

which ensures that the series for the root rapidly converges if

$$|\theta| = \vartheta < 1.$$

Of course, the actual domain of convergence is larger than the domain defined by the zeroth sufficient condition. For example, for the equation

$$y^2 + y - 1 = 0,$$

defining a golden section, we have

$$a = -1, \quad b = 1, \quad c = 1,$$

and the zeroth sufficient condition for convergence is not fulfilled, because

$$\theta = \frac{4ac}{b^2} = -4,$$

Nevertheless, the result even at the first iteration step falls into the unit disk:

$$\theta_1 = 0.444444444,$$

where the first sufficient condition for convergence holds, and the subsequent iterates decrease rapidly:

$$\theta_2 = 0.081632888,$$

$$\theta_3 = 0.001810774,$$

$$\theta_4 = 0.000002050,$$

$$\theta_5 = 0.000000000.$$

CONSTRUCTION OF A ZERO APPROXIMATION

Consider the Airy equation

$$\psi'' + x\psi = 0.$$

Applying the change of variables

$$S = -\frac{\psi'}{\psi},$$

which goes back to d'Alembert [2], we obtain the Riccati equation

$$S' = x + S^2.$$

In this form, the equation represents the worst case for constructing an iterative process. It contains no linear term and, hence, formally speaking, $\theta = \infty$. For this reason, we have to make one more change of variables

$$S = q + \frac{1}{m} y,$$

where q is an arbitrary (complex) function and m is a positive function:

$$my' = a + by + y^2.$$

The coefficients $a(x)$ and $b(x)$ are determined by choosing a zero approximation $q(x)$ and an arbitrary positive function $m(x)$:

$$a(x) = m^2(x + q^2 - q'), \quad b(x) = 2mq + m'.$$

The key point is the choice of $q(x)$. The most successful is an idea going back to Euler. Specifically, $q(x)$ is defined parametrically in the complex plane z:

$$x = -z^2 + \frac{A}{z}, \quad q = z + \frac{B}{z^2}.$$

Simple but cumbersome calculations give

$$a(x) = m^2$$
$$\times\left[B^2z^{-4} + \frac{A^2 + 2BA - 2B + (2A + 4B + 1)z^3}{z(2z^3 + A)}\right],$$

$$b(x) = 2m\left(z + \frac{B}{z^2}\right) + m'.$$

The coefficient $a(x)$ can be considerably reduced by setting the second term equal to zero. This produces two equations

$$A^2 + 2BA - 2B = 0, \quad 2A + 4B + 1 = 0.$$

The system of equations solved for A and B gives

$$A = -1, \quad B = 1/4.$$

Finally,

$$x = -z^2 - \frac{1}{z}, \quad q = z + \frac{1}{4z^2},$$

$$a(x) = \frac{m^2}{16z^4},$$

$$b(x) = 2m\left(z + \frac{1}{4z^2}\right) + m',$$

$$p(x) = 2m\mathrm{Re}\left(z + \frac{1}{4z^2}\right) + m'.$$

Changing to polar coordinates in the plane z, we obtain

$$z = r\exp(i\varphi),$$

$$a(x) = \frac{m^2}{16r^4},$$

$$p(x) = \frac{m}{2r^2}(4r^3\cos\varphi + 2\cos^2\varphi - 1) + m'.$$

Separating the real part in the formula for x,

$$x = -z^2 - \frac{1}{z},$$

we have

$$x = -r^2\cos 2\varphi - \frac{1}{r}\cos\varphi,$$

$$0 = -r^2\sin 2\varphi + \frac{1}{r}\sin\varphi.$$

The second equation has two branches: horizontal

$$\sin\varphi = 0$$

and vertical

$$-2r^2\cos\varphi + \frac{1}{r} = 0.$$

These branches intersect at the branch point

$$\varphi_0 = 0, \quad 2r_0^3 = 1.$$

The vertical branch is defined by the equation

$$2r^3\cos\varphi - 1 = 0.$$

At the branch point, it has a vertical tangent and asymptotically approaches the ordinate axis. The calculation of $|a(x)|$ and $p(x)$ gives

$$|a(x)| = \frac{m^2}{16r^4},$$

$$p(x) = \frac{m}{2r^2}(4r^3\cos\varphi + 2(\cos\varphi)^2 - 1) + m'.$$

On the vertical branch, it is natural to set

$$m = r^2.$$

Then, all quantities required for estimation are functions of r alone, because the trigonometric functions can be eliminated since the equation of the vertical branch implies the relation

$$\cos\varphi = \frac{1}{2r^3}.$$

As a result, we have

$$|a(x)| = \frac{1}{16}, \quad p(x) = \frac{1}{4}\frac{5r^6 + 6r^{12} + 2}{r^6(2 + r^6)},$$

and $p_v = \min_x p(x)$ is now easy to find. On the horizontal branch, it is equal to the value of $p(x)$ at the branch point, the leftmost point of the horizontal branch:

$$p_h = 1.5.$$

Since

$$p = \min(p_v, p_h) = p_v = 1.082106781,$$

we finally obtain for the zero approximation

$$a = \frac{1}{16}, \quad p = 1.082106781,$$

and, hence,

$$\theta = \frac{4a}{p^2} = 0.213500930.$$

The iterative process rapidly converges, and the third iterate gives nine correct digits:

$$\theta_0 = 0.213500930,$$
$$\theta_2 = 0.014282161,$$
$$\theta_3 = 0.000000000.$$

CONCLUSIONS

We have constructed a series expansion of the solution to the Riccati equation bounded on the entire line. The terms of this series are bounded solutions to linear equations. The series converges absolutely and uniformly on the entire line, and the convergence rate is the same as in Newton's method; i.e., the number of correct digits is doubled at every iteration step.

The analysis conducted here is different from traditional considerations. We analyzed the Riccati equation [2] rather than the Sturm–Liouville equation and sought a complex particular solution [9, 10] rather than a real general one. A convergent iterative process was constructed instead of an asymptotical series [6–8]. The zero approximation was specified parametrically rather than explicitly. Following Max Planck, the zero approximation was constructed as a whole, not locally.

The Airy equation describes the borderline of geometric optics (specifically, it describes the behavior of a quantum particle near a turning point [3–5]).

Traditionally, estimates in absolute value are typical of relaxation processes. Oscillating systems exhibit substantial interference, and, thus, estimates in absolute value can hardly be obtained. Nevertheless, we showed here that an absolutely and uniformly convergent iterative process can be constructed by considering the problem in the complex domain and by picking a reasonable zero approximation.

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